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On the stability of the motion of uncharged dipoles

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Abstract. We examine the stability of the repulsive motion of an uncharged dipole along the axis of symmetry of a stationary dipole field. Then we let the dipole have a large initial spin about its axis of symmetry, establish the equations of motion in the external field, find three integrals, and construct the trajectory of the system. Finally, we allow the dipole to make a certain angle with the axis of symmetry of the mass distribution, at the same time separating the motion of the dipole from the fast rotations of the mass distribution. The dipole's direction may be varied by inner forces. By this, the device becomes dirigible.

1. Introduction

The uncharged dipole can often be used to gain an approximate description of the behaviour of neutrons, dipole moleculaes, magnetic coils or solenoids. The motion of a dipole is more complicated than the motion of a point charge because two additional variables are necessary to fix the orientation of the dipole's axis of symmetry. In an inhomogeneous external field, a small variation of the initial values of these variables can lead to a totally different motion after a short period of time. The interaction of the dipole with the external field is described by an energy expression of the form $-\boldsymbol{MH}$, where $\boldsymbol{M}$ is the dipole moment and $\boldsymbol{H}$ the field strength. For the external field, we take the field of a finite current or charge distribution (with total charge being zero) at distances large in comparison with the dimensions of the distribution, i.e. the dipole field.

To examine the deflection of a trajectory for small changes of the initial data, we choose the simplest trajectory of the dipole in the dipole field, i.e. the infinite rectilinear motion along the symmetry axis of the dipole field. The corresponding initial data are such that the axis of the dipole lies exactly on the axis of symmetry of the external field and the dipole is repelled. This motion is unstable in two ways: if the two axes of symmetry are not exactly parallel, the dipole begins to rotate and forces arise that move it from the axis of the external field; if the dipole does not lie exactly on the axis of the external field, it also moves away from it and begins to perform a complicated rotational motion (see § 2).

A large initial spin about the axis of the dipole will prevent the dipole from rotating about the axes perpendicular to it (§ 3). The external couple will now lead to small nutations of the dipole axis. The motion of the dipole axis is however more complicated than in the case of the symmetric heavy top or of the dipole spinning in a homogeneous field, since it gyrates in a field that is not simply homogeneous. We establish the equations of motion and find three distinct integrals in § 3.3, which enable us to evaluate the trajectory of the system.

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Once the direction of the dipole has been stabilised, the dipole is repelled with a maximum gain of energy. Small perturbations in its exact position on the symmetry axis of the external field and in the direction of its velocity will, however, generate forces that move the mass centre away from this axis. To obtain compensating forces, we allow (in § 4) for the possibility that the dipole can make an angle with the direction of spin, the angle being controlled by inner forces. At the same time we separate the motion of the dipole from the fast rotation of the fly-wheel. The dipole then ceases to be subjected to great centrifugal forces. It is shown that the two inner degrees of freedom can be used to pilot the device in a relatively aribtrary way.

A note by Engelberger (1964) is partly related to § 2. The author has recently applied § 2.2 in astronautics (Lemke 1981).

1.1. Basic equations and conventions

The motion of the dipole depends on the mass distribution. In principle, the dipole can lie at any point of the rigid body. We will confine ourselves to the simple, though realistic, case where two centroidal principal moments of inertia are equal and the dipole rests in the mass centre. In § 2 and § 3 we let the dipole axis coincide with the third axis of the momental ellipse, as is suggested by the symmetry of a dipole field.

Let $Oxyz$ be a right-handed system of rectangular axes fixed in the external field. The origin is taken to coincide with the centre of the field and the z axis with its axis of symmetry. In this system we use cylindrical coordinates z, r, ϕ to determine the position of the mass centre of the dipole. These coordinates take into account the axial symmetry of the external field (H_ϕ is zero and H_r and H_z do not depend on ϕ). In these coordinates the energy of translation is

$$T_{\text{trans}} = \tfrac{1}{2}m(\dot{z}^2 + \dot{r}^2 + r^2\dot{\phi}^2) \tag{1.1}$$

where m is the total mass of the dipole.

Let $O'XYZ$ be another system of rectangular axes parallel to the respective axes of $Oxyz$ but with origin O' located at the mass centre of the dipole. In this coordinate system, Euler angles (θ, φ, ψ) define the position of the body, where θ is the inclination of the third principal axis to the Z axis and φ is the direction of line of nodes in the XY plane. The kinetic energy of motion relative to mass centre is then given by

$$\mathrm{T}_{\text{rot}} = \tfrac{1}{2}I(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2\theta) + \tfrac{1}{2}I_3(\dot{\varphi}\cos\theta + \dot{\psi})^2 \tag{1.2}$$

where I_3 is the axial moment of inertia and I is the transverse moment of inertia at the mass centre.

Let θ be measured up from the upward vertical so that the dipole may be parallel to the dipole moment of the external field for $\theta = 0$ (it is then attracted). Moreover, let $\varphi = 0$ and $\phi = 0$ mean the same direction, say the positive X direction. The interaction term will then take the following form:

$$\boldsymbol{M}(H_z\boldsymbol{e}_z + H_r\boldsymbol{e}_r) = MH_z(z, r)\cos\theta + MH_r(z, r)\sin\theta\sin(\varphi - \phi) \tag{1.3}$$

where $\phi - \varphi + 90°$ is the angle between the projection of $\boldsymbol{M}$ and $\boldsymbol{H}(z, r)$ on a plane $z = \text{constant}$. The dipole field is given by

$$H_z = \bar{m}(2z^2 - r^2)/R^5, \qquad H_r = \bar{m}3zr/R^5, \tag{1.4}$$

where $\bar{m}$ is the dipole moment of the external field and $R^2 = z^2 + r^2$. The Lagrange function L of our system is $T_{\text{trans}} + T_{\text{rot}} + \boldsymbol{MH}$ which yields the equations of motion analysed in § 2 and § 3.

Let z_0, the initial z value, be the typical length scale, and let us measure z, r and R in units of z_0. Moreover, we divide the Lagrangian by mz_0^2 and define $a^2 \equiv I/(mz_0^2)$ and $b \equiv I_3/I$ so that 'a' and 'b' are dimensionless, and we let $g \equiv 6M\bar{m}/(mz_0^5)$ be a z_0-dependent 'coupling constant' (dimension: time^{-2}).

2. Zero initial spin

The I_3 term of T_{rot}, equation (1.2), is assumed to be zero at the beginning of the motion; it remains zero because it is a constant of the motion. At first, we examine the infinite rectilinear motion, for which $\theta \equiv +\pi$ and $r \equiv 0$. Then we consider the case where these identities are initially slightly perturbed.

2.1. *The infinite rectilinear motion*

For $\theta \equiv +\pi$ and $r \equiv 0$ the Lagrange function is very simple. It describes a rectilinear translatory motion of the mass centre along z. The equation of motion is $\ddot{z} = g/z^4$. It shows that the acceleration of the dipole decreases rapidly with increasing z if an increase of g, i.e. M or $\bar{m}$, does not compensate the z^{-4} fall-off. We shall, however, take g as time independent. Thus, the motion becomes quickly practically uniform. If the dipole has initially the velocity $\dot{z}_0$, $\dot{z}(z)$ is given by

$$\dot{z}^2 = \tfrac{2}{3}g(1 - z^{-3}) + \dot{z}_0^2 \tag{2.1}$$

so that

$$\dot{z}_\infty^2 = \tfrac{2}{3}g + \dot{z}_0^2. \tag{2.2}$$

Already at the position $z = 2$, velocity $\dot{z}$ is only about 6% smaller than $\dot{z}_\infty$ if $\dot{z}_0 = 0$, and this deficit is less if $\dot{z}_0 > 0$. The law of motion, $z(t)$ or $t(z)$, cannot be expressed by elementary functions (see e.g. theorem of Tshebyshev). Let us however evaluate a characteristic time interval of the motion, e.g. the time t_2 in which z doubles. We find

$$t_2 \sim (3/2g)^{1/2}\,\tfrac{4}{3} \tag{2.3}$$

for $\dot{z}_0 = 0$, where the $\frac{4}{3}$ is an approximate value.

2.2. *The case of the rigid dipole axis*

If the dipole in the initial position $\theta_0 = \pi$ does not lie exactly on the external field's symmetry axis but has a small $r_0 \neq 0$, a small force F_r is present. This moves the dipole away from the axis of symmetry as z increases, giving a plane curvilinear translatory motion. Let us consider the case where $\dot{\theta}$ remains zero, that is, where the dipole keeps its initial orientation $\theta_0 = \pi$. This case is reached in the limit $a \to \infty$, i.e. for a mass distribution of very great dimensions, or where the θ degree of freedom is well stabilised by a spin angular momentum in the position $\theta \sim \pi$ (see § 3).

The Lagrange function gives us the equations of motion

$$\ddot{z} \equiv f_z = g(z^3/R^7)(1-\tfrac{3}{2}r^2/z^2), \qquad \ddot{r} \equiv f_r = g(2rz^2/R^7)(1-\tfrac{1}{4}r^2/z^2). \tag{2.4}$$

On lines given by $z=(3/2)^{1/2}r$, f_z disappears, and on the less steep lines $z=\frac{1}{2}r$, f_r disappears. We will only deal with a small initial coordinate $r_0 \ll 1$. Because f_r/f_z is small (of the order of r_0), r/z will remain small, at least in the region $1 \leqslant z \leqslant 2$. At the larger values of z, the forces are so weak (as we saw in § 2.1) that they can change the momentum that the field has already imparted to the dipole only slightly. This means the trajectory remains in the sector $r/z \ll 1$ and we can expand (2.4) in terms of small r/z.

In the following $\dot{z}_0 = \dot{r}_0 = 0$. The equation of the trajectory $r(z)$ is then given by

$$\frac{dr}{dz} = \frac{f_r}{f_z} - \frac{2}{f_z}\frac{d^2r}{dz^2}\frac{U_0-U}{1+(dr/dz)^2} \tag{2.5}$$

where U is the potential energy, $U=\frac{1}{6}g(2z^2-r^2)/R^5$, and $\dot{z}^2$ has been eliminated by the energy integral. Initially, U_0-U is zero and thereafter small, so that the second term can be neglected; thus the initial trajectory follows the force line through (r_0,z_0). This means that d^2r/dz^2 is positive, so that at the larger values of z, where the second term must be taken into account, dr/dz is less than f_r/f_z. That is, the trajectory is steeper than the force lines that it intersects. For $z \sim 2$ the trajectory is near to the asymptotic region, where it is practically straight. The value of $(dr/dz)_\infty$ is a characteristic quantity of the trajectory. To find its dependence on r_0 (and z_0) let us rewrite (2.5) in the form

$$d^2r/dz^2 = (f_r - f_z dr/dz)/\dot{z}^2. \tag{2.6}$$

The second derivative and the numerator of the RHS are small quantities. Hence, $\dot{z}^2$ can be replaced by its zero-order value (2.1):

$$\frac{d^2r}{dz^2} = \frac{3}{2z^2}\frac{2r - z\,dr/dz}{z^3-1}. \tag{2.7}$$

Since d^2r/dz^2 is positive definite, $2r > z\,dr/dz$ which means that r cannot increase more strongly than z^2. This shows that d^2r/dz^2 decreases more strongly than z^{-3}. Hence

$$\frac{dr(z)}{dz} = \int_1^z \frac{d^2r}{dz^2}dz + 2r_0 \tag{2.8}$$

very nearly equals its asymptotic value at z in the range of 2 to 3. For the evaluation of the integral it is thus important to know the second derivative as accurately as possible at $z-1$. Expanding $r(z)$ in terms of small $z-1$ and comparing the coefficients in (2.7) yields $r=r_0+2r_0(z-1)+\frac{1}{2}r_0(z-1)^2+\ldots$. We substitute this result into the RHS of (2.7) and find $d^2r/dz^2 \sim 2r_0/3z^2$. This approximation is exact for $z \to 1$. Therefore, it gives us the exact value of dr/dz_∞ because the function $\int^z dz\, d^2r/dz^2$ disappears for $z \to \infty$. Substitution into (2.8) yields

$$dr/dz_\infty = 2r_0(1+\tfrac{1}{6}) \tag{2.9}$$

which shows that the initial inclination of the trajectory only increases by $\frac{1}{6}$. As this increase is rather small, our approximations are valid so long as equation (2.7) is a good approximation for equation (2.6) at (z_0, r_0). This is the case with an accuracy better than 20% for $r_0 < \frac{1}{5}$†.

2.3. The true motion

A small r_0, $|\pi - \theta|_0$ or $\dot{\theta}_0$ is sufficient to cause large changes in θ while the dipole is pushed away to large z. Angle θ will pass $\pi/2$, 0 and then even $-\pi$ because the forces acting on the dipole in the second half period $0 \geq \theta \geq -\pi$ are weaker than those acting in the first half period $\pi \geq \theta \geq 0$ at the beginning of the motion. That is, the dipole will perform rotational oscillations. The θ changes are the faster the greater g (the external force) and the smaller a (the transverse radius of the body). If $\theta_0 = \pi$ and $\dot{\theta}_0 \leq 0$, we have $\sin(\varphi - \phi) = 1$ automatically at the beginning of the motion, since this is the minimum point of the potential energy. After θ passes $\theta = 0$, the motion in φ and ϕ however becomes unstable and rather complicated. Let us neglect the resulting small motions in φ and ϕ. The dipole's trajectory $z(r)$ will then oscillate about a path that is similar to that described in § 2.2. Asymptotically, the dipole will move under a certain positive inclination $\mathrm{d}r/\mathrm{d}z_\infty$ to the z axis and possess a certain angular momentum $L = I\dot{\theta}_\infty$ about the mass centre.

3. Large initial spin—behaviour in $\theta\varphi\phi$ space

To stabilise the direction of the dipole in the position in which it points vertically downward, we suppose the body to have a large angular velocity Ω_3 about its axis of symmetry,

$$\Omega_3 = \dot{\varphi} \cos \theta + \dot{\psi}.$$

θ will then only vary in a small interval near $\theta \sim \pi$ and can be called the angle of nutation. As ψ is an ignorable variable, Ω_3 is a constant of the motion. (Thus the time dependence of $\dot{\varphi}$ and $\cos\theta$ give the time dependence of $\dot{\psi}$.)

3.1.

Because r_0 and $\pi - \theta$ are small, we can, in a first approximation, neglect H_r, so that φ, too, becomes an ignorable variable (approximation of parallel field lines). The canonical momentum integral conjugate to φ is

$$\dot{\varphi} \sin^2 \theta + b\Omega_3 \cos \theta = -b\Omega_3 \equiv -2\omega. \tag{3.1}$$

(The third axis is assumed to be antiparallel to the z axis at time $t = 0$.) Using these equations, we eliminate $\dot{\varphi}$ in the energy integral, which gives us the following effective

† In a numerical analysis of (2.7) it may be helpful to know that (2.7) can be integrated exactly once in the interesting region of small $z - 1$. In this region we have $z^3 - 1 = 3(z-1) + 3(z-1)^2 + (z-1)^3 \sim 3z(z-1)$ with good accuracy up to $z \sim 1.6$. Moreover, (2.7) is a first-order differential equation for $y(z) \equiv r^{-1}\,\mathrm{d}r/\mathrm{d}z$ with the initial condition $y(1) = 2$. Approximating $z^3 - 1$ in the way indicated, (2.7) gives us a Riccati equation for y that can be solved by elementary functions. The solution is

$$y = [1/(z-1)]\{1 - \tfrac{1}{2}z^{-2} - \tfrac{1}{2}z^{-1}[1 + (\pi e)^{1/2} s\, \mathrm{e}^{-1/2z} \operatorname{erf} s]^{-1}\}$$

where $s^2 = \frac{1}{2}(1 - z^{-1})$. This solution even exhibits the right asymptotic behaviour $y \sim 1/z$, i.e. $\mathrm{d}r/\mathrm{d}z =$ constant.

potential energy at small $\pi-\theta\equiv\bar{\theta}$:

$$\tfrac{1}{2}a^2\dot{\varphi}^2\bar{\theta}^2-\tfrac{1}{2}a^2h_z\bar{\theta}^2=\tfrac{1}{2}a^2(\omega^2-h_z)\bar{\theta}^2,$$

where $h_z\equiv MH_z/I$ has been defined. This expression is least at $\bar{\theta}=0$ provided $\omega^2>h_z$, which is the condition of stability for the θ degree of freedom in the position $\theta_0=\pi$. As h_z decreases with time, this condition is always satisfied provided it is satisfied at $t=0$.

For small initial $\dot{\theta}_0$, the variable θ oscillates harmonically about the value $\theta=\pi$. The frequency is given by

$$\omega_0^2=\omega^2-h_z. \tag{3.2}$$

The amplitude is of order $\dot{\theta}_0/\omega_0$. Let us use this expression to compare the periodic time $2\pi/\omega_0$ with the characteristic time $t_*\equiv 1/\sqrt{g}$ of the change in h_z (see (2.3)). If the rotational energy $\frac{1}{2}I_3\Omega_3^2$ is twice as large as the magnetic energy MH_0, then

$$\omega_0t_*=(b-1)^{1/2}/(a\sqrt{3}). \tag{3.3}$$

$a\sqrt{3}$ is just the transverse dimension of the body, which in practice will be much smaller than 1 (i.e. z_0). Hence, ω_0^{-1} will be much smaller than t_* even if the rotational energy is only twice as large as the magnetic energy.

3.2. *The equations of motion*

Let us take into account the H_r term. (3.1) is no longer a constant of the motion. As, however, $f_\varphi+f_\phi=0$, there is an analogous constant. Fixing its value by the initial data $\theta_0=\pi$, $\dot{\phi}_0=0$, we find

$$\dot{\phi}=(a^2/r^2)\bar{\theta}^2(-\dot{\varphi}-\omega). \tag{3.4}$$

The sum $\dot{\varphi}+\omega$ is of the order of magnitude of ω, a high frequency, while, as we will see later on, the factor in front of the sum is much less than one. Hence $\dot{\phi}$, the angular velocity of mass centre about the z axis, is small compared with ω.

Let us write down the Lagrangian equations of motion for θ, φ and ϕ:

$$\ddot{\bar{\theta}}=(\dot{\varphi}^2+2\omega\dot{\varphi}+h_z)\bar{\theta}+h_r\sin(\varphi-\phi), \tag{3.5a}$$

$$\mathrm{d}[(\dot{\varphi}+\omega)\bar{\theta}^2]/\mathrm{d}t=h_r\bar{\theta}\cos(\varphi-\phi), \tag{3.5b}$$

$$\mathrm{d}(r^2\dot{\phi})/\mathrm{d}t=-h_ra^2\bar{\theta}\cos(\varphi-\phi),\qquad \text{where } h_r\equiv MH_r/I. \tag{3.5c}$$

The analysis of this system of three second-order differential equations is a complex task. First, let us establish the time dependence of the solution at the beginning of the motion for $\bar{\theta}_0=0$, $\dot{\phi}_0=0$ and $\dot{\theta}_0=0$. We find

$$\varphi_0-\phi_0=+\pi/2 \tag{3.6a}$$

so that $\ddot{\bar{\theta}}$, i.e. $\bar{\theta}$, is positive (see (3.5a)). The value (3.6a) has already appeared in § 2, as the minimum of the potential energy in $\varphi-\phi$. It now follows from (3.5a) that $\bar{\theta}_1=\frac{1}{2}h_rt^2$, which we use in (3.5$b$) to find that

$$\dot{\varphi}_0=-\tfrac{2}{3}\omega. \tag{3.6b}$$

Substituting this result into (3.4) or (3.5c), we can find $\dot{\phi}_1$, which will be much smaller than $\dot{\varphi}_0$. Define $\varphi-\phi\equiv\alpha$. Then $\alpha=\pi/2-\frac{2}{3}\omega t+\ldots$ which gives us

$$\alpha(\theta)=\tfrac{1}{2}\pi-\tfrac{2}{3}\omega(2\bar{\theta}/h_r)^{1/2}+\ldots \tag{3.7}$$

for the trajectory $\alpha(\theta)$. At the beginning of the motion, ϕ practically does not change while α (and φ) decrease as the negative of the square root of $\bar{\theta}$.

3.3. Integrals

We now consider the $\theta\varphi\phi$ motion over a period of time Δt, $t_* \gg \Delta t \gg \omega_0^{-1}$, where r and z can be taken as constant (the greater ω_0, the closer this approximation will be, see § 3.1) and derive the energy integral of the set of equations (3.5) from L. One finds that

$$(a/r)^2\bar{\theta}^4(\dot{\varphi}+\omega)^2+\dot{\theta}^2+(\dot{\varphi}^2-h_z)\bar{\theta}^2=2h_r\bar{\theta}\sin\alpha \qquad (3.8)$$

if one makes use of the initial data $\dot{\phi}_0=\dot{\theta}_0=\bar{\theta}_0=0$ and eliminates $\dot{\phi}$ by (3.4). The equation shows that $\bar{\theta}$ must be finite. $\bar{\theta}_{max}$ is obviously reached if the fourth-order $\bar{\theta}$ term dominates the second-order $\bar{\theta}$ term. Then $\bar{\theta}$ reaches its maximum for $\sin\alpha = 1$, $\dot{\theta}=0$ and $\dot{\varphi}=0$. We find

$$\bar{\theta}^3_{max}=(2h_r/\omega^2)(r/a)^2 \qquad \dagger. \qquad (3.9)$$

If, however, $\dot{\varphi}$ was able to reach the value $-\omega$, the second-order term would dominate and $\bar{\theta}_{max}$ would have the value of $2h_r/\omega^2$. Even for a lower value of ω^2 which is given by $\omega^2 \sim h_z$, this $\bar{\theta}_{max}$ is small of the order of $2h_r/h_z$, whereas $\bar{\theta}_{max}\sim(2h_r r^2/h_z)^{1/3}a^{2/3}$ according to (3.9), which number, as we saw under (3.3), is indeed larger. This number must, of course, be much smaller than one for the expansion in terms of small $\bar{\theta}$ to remain applicable.

Equations (3.4) and (3.8) are two integrals in the three-dimensional $\theta\varphi\phi$ space. If we were able to find a third integral, the trajectory of the system could be constructed. Let us calculate the second time derivative of $\bar{\theta}\cos\alpha$ and eliminate $\ddot{\theta}$ and $\ddot{\varphi}$ by (3.5):

$$\frac{d^2}{dt^2}(\bar{\theta}\cos\alpha)=2\omega\frac{d}{dt}(\bar{\theta}\sin\alpha)+\frac{d}{dt}(\bar{\theta}\dot{\phi}\sin\alpha)+(2\omega\dot{\phi}+h_z)\bar{\theta}\cos\alpha$$
$$+\dot{\bar{\theta}}\dot{\phi}\sin\alpha+\bar{\theta}\dot{\varphi}\dot{\phi}\cos\alpha.$$

Because of (3.5*c*), the third term, too, is, in the approximation of constant r and z, a time derivative. Furthermore, the last two terms can be neglected in relation to the first term, i.e. it can be shown that always $|\dot{\phi}| \ll \omega$. Let us give a simple proof. Obviously, $|\dot{\varphi}|$ will not be larger than of order ω. Together with (3.9) this means that, in (3.4),

$$|\dot{\phi}|<(2h_r/h_z)^{2/3}(a/r)^{2/3}\omega \qquad \text{for } \omega^2 \text{ as small as } h_z, \qquad (3.10)$$

the factor of ω being indeed small compared with 1, as we saw under (3.3). (Note: $h_r/h_z=O(r)$.)

By this, we possess a third integral, which disappears for our initial data $\bar{\theta}_0=\dot{\theta}_0=\dot{\phi}_0=0$. We can simplify their form by introducing a dimensionless time τ and a natural $\bar{\theta}$ variable ϑ:

$$t=\frac{1}{2\omega}\tau, \qquad \bar{\theta}=(h_r/2\omega^2)\vartheta$$

(note that ϑ_{max} can be much larger than one). Rewriting the integrals in the new

† If the rotational energy is sufficiently great, this expression will also give the approximate time dependence of $\bar{\theta}_{max}$. As ω^2 is a constant of the motion, $\bar{\theta}_{max}$ changes like $r/z^{4/3}$ with time.

variables yields

$$(1/P)\dot{\phi}^2+\dot{\vartheta}^2+(\dot{\varphi}^2-\tfrac{1}{2}Q)\vartheta^2=\vartheta\sin\alpha, \tag{3.11a}$$

$$(1/P)\dot{\phi}=-\vartheta^2(\dot{\varphi}+\tfrac{1}{2}), \tag{3.11b}$$

$$\dot{\vartheta}\cos\alpha-\vartheta(\dot{\varphi}+1)\sin\alpha=-(1/P)\dot{\phi}^2-(Q/P)\dot{\phi}, \tag{3.11c}$$

where the derivatives are now with respect to τ, and

$$P\equiv h_r{}^2a^2/4\omega^4r^2, \qquad Q\equiv h_z/2\omega^2.$$

The case of a large spin corresponds to $Q\ll 1$. The evaluation (3.10) has shown that $\bar{\theta}^2a^2/r^2\ll 1$. In terms of ϑ this is the same as

$$P\vartheta^2\ll 1. \tag{3.10$'$}$$

Instead of the second-order system (3.5) we can now examine the first-order system (3.11). The field of the gradients ($\mathrm{d}\vartheta$, $\mathrm{d}\varphi$, $\mathrm{d}\phi$) in the $\vartheta\varphi\phi$ space depends only on ϑ and $\alpha=\varphi-\phi$; it does not depend on $\varphi+\phi$, the $(\varphi+\phi)$ axis being perpendicular to the $\vartheta\alpha$ plane. Studying the field of inclinations $\mathrm{d}\alpha/\mathrm{d}\vartheta$ will thus provide a good idea of the form of the solution trajectory, which is a monotone, single-valued function of the length of its projection on the $\vartheta\alpha$ plane (the function being defined on the cylindrical plane perpendicular to the $\vartheta\alpha$ plane through this projection).

3.4. *Trajectories*

Let us first examine how the trajectory (3.7) continues in the region $\vartheta\ll 1$. Because of (3.10$'$), we can neglect the $\dot{\phi}$ term in (3.11a)—it is much smaller than the RHS in our case of $\alpha_0=\pi/2$. Equally well, the Q term can be omitted, and also the $\dot{\phi}$ terms in (3.11c). The ϕ dependence of the trajectory is thus negligible even for values of ϑ of the order of 1; ϕ can be regarded as constant. The remaining two equations can easily be solved:

$$\dot{\vartheta}=\vartheta(\dot{\varphi}+1)\tan\alpha, \tag{3.12a}$$

$$\dot{\varphi}=-\sin^2\alpha+\cos\alpha(\vartheta^{-1}\sin\alpha-\sin^2\alpha)^{1/2}. \tag{3.12b}$$

Substituting expansion (3.7) into (3.12b), one can see that $\dot{\varphi}(\vartheta)$ reproduces the value (3.6b), i.e. $\dot{\varphi}_0=-\frac{1}{3}$. The trajectories cannot pass the cylindrical plane

$$\vartheta^{-1}=\sin\alpha, \tag{3.13}$$

which plane comes from infinite ϑ at $\alpha=\pi$ and departs to infinite ϑ at $\alpha=0$. Moreover, the intersection of this plane with the $\vartheta\varphi$ plane is a trajectory, that is, this intersection satisfies equations (3.12). On this trajectory we have $\dot{\varphi}=-\sin^2\alpha$ and $\dot{\vartheta}=\cos\alpha$. A further characteristic structure of the gradient field $\mathrm{d}\varphi/\mathrm{d}\vartheta$ is the curve

$$\vartheta=1/\sin\alpha-\sin\alpha, \qquad 0\leqslant\alpha\leqslant\pi/2 \tag{3.14}$$

on which $\dot{\varphi}=0$ and $\dot{\vartheta}=\cos\alpha$. For $\pi\geqslant\alpha\geqslant\pi/2$, $\dot{\varphi}$ is negative definite and does not vanish. A comparison of curve (3.14) with trajectory (3.7) shows that the trajectory lies at a somewhat larger α. The gradient field pertaining to system (3.12) is depicted in figure 1(a) together with the other results. One sees that the solution passes between curves II and III to large ϑ into a region of small $|\dot{\varphi}|\ll 1$. Our approximation is no longer valid in this region; the $\dot{\phi}$ terms begin to play an important role.

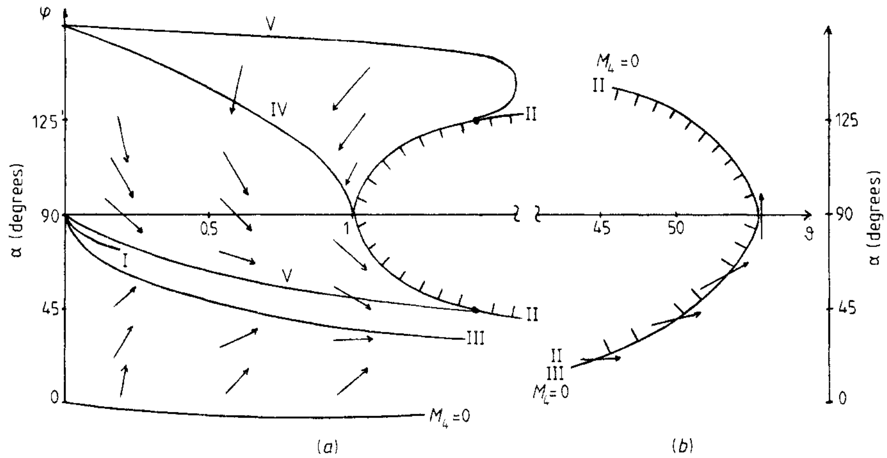

Figure 1. (a) Gradient field in a plane $\phi =$ constant, in which the solution trajectory approximately lies for small $\vartheta \leqslant 1$. I is the solution at the beginning; II is the curve $\vartheta = 1/\sin\alpha$ which is a boundary of the kinematic region and also a trajectory ($\dot\vartheta = \cos\alpha, \dot\varphi = -\sin^2\alpha$, $\dot\phi = -P\vartheta^2(\frac{1}{2} - \sin^2\alpha)$); III is the curve $\dot\varphi = 0$, $\dot\vartheta = \cos\alpha$, $\dot\phi = -P\vartheta^2/2$; IV is the curve $\dot\vartheta = 0$, $\dot\varphi = -1$, $\dot\phi = +P\vartheta^2/2$; V is the curve $\dot\phi = 0$, $\dot\varphi = -\frac{1}{2}$, $\dot\vartheta = \frac{1}{2}\vartheta\tan\alpha$. ($b$) Form of curve III ($M_4 = \cos^2\alpha$) at large ϑ for $P/4 = 10^{-5}$ and $Q = \frac{1}{40}$. The inner kinematic limit (II) lies only a little to the left of curve III, and touches it in its end point. The arrows indicate the inclinations of the trajectories on curve III. On curve $M_4 = 0$, we have $\dot\varphi = 0$ and $\dot\vartheta = 0$.

Let us look for the continuation of line III in figure 1(a). If one subtracts (3.11c) from (3.11a) and eliminates $\dot\phi$ by means of (3.11b) one finds the equation

$$\dot\vartheta(\dot\vartheta - \cos\alpha) = -\vartheta^2\dot\varphi[\dot\varphi + (\sin\alpha)/\vartheta + Q]. \tag{3.15}$$

It shows that there is a curve (plane) $\dot\varphi = 0$, $\dot\vartheta = \cos\alpha$ not only for values of $\vartheta \leqslant 1$ but for all values of ϑ. This curve has the equation

$$\cos^2\alpha = \vartheta\sin\alpha + \tfrac{1}{2}Q\vartheta^2 - \tfrac{1}{4}P\vartheta^4 \equiv M_4, \qquad 0 \leqslant \alpha \leqslant \pi/2, \tag{3.14$'$}$$

and reaches its largest ϑ value in its end point $\vartheta \equiv \vartheta_1$, $\alpha = \pi/2$, the value being given by about $\vartheta_1 \sim (4/P)^{1/3}$ (which is exactly the same as (3.9)). The curve is shown in figure 1(b).

The trajectories' inclinations on curve (3.14′) follow from (3.11b) and read

$$\mathrm{d}\alpha/\mathrm{d}\vartheta = \tfrac{1}{2}P\vartheta^2/\cos\alpha, \tag{3.16}$$

a positive definite function. It is to be compared with the $\mathrm{d}\alpha/\mathrm{d}\vartheta$ of the plane (3.14′). By differentiation of (3.14′) one finds that in the large-ϑ region, $1 \ll \vartheta < \vartheta_1$, $\mathrm{d}\alpha/\mathrm{d}\vartheta$ of curve (3.14′) is larger than the inclination (3.16) of the trajectories on this curve (see figure 1(b)). Furthermore, the inner kinematic limit will touch curve III at the end point $\vartheta = \vartheta_1$, $\alpha = \pi/2$ because here $\dot\vartheta = 0$†. Consequently, *all* the trajectories between the kinematic limit II and curve III in figure 1(a) will intersect the cylindrical plane (3.14′) at large ϑ.

Let us now look for the cylindrical planes on which $\dot\vartheta = 0$. Equation (3.15) shows that we have either $\dot\varphi = 0$ or $\dot\varphi = -(\sin\alpha)/\vartheta - Q$ on such planes. The first solution

† If one derives the equation of the inner kinematic limit for large ϑ and $\alpha \sim \pi/2$, one will find an equation of the type $M_4 = C\cos^2\alpha$, where C is a function of P and Q.

yields the plane $M_4 = 0$ in (3.11*a*) or (3.11*c*). This plane lies somewhat to the right of plane III and also touches it at the maximum $\vartheta = \vartheta_1$, $\alpha = \pi/2$. Left of the plane $M_4 = 0$, $\alpha \leqslant \pi/2$, we have $\dot{\vartheta} > 0$, whole right of this plane and above the inner kinematic bound in the upper half-plane $\alpha \geqslant \pi/2$ we have $\dot{\vartheta} < 0$ up to curve IV in figure 1(*a*).

Consequently, all the trajectories between plane (3.14′) and $M_4 = 0$ (among which is the solution trajectory) pass the perpendicular in the end point of curve (3.14′). On this perpendicular they have $\dot{\vartheta} = 0$, that is, they reach their largest ϑ value $\vartheta = \vartheta_1$, the same for all these trajectories. Between these planes $\dot{\varphi}$ is positive, that is, $\varphi(t)$ increases here for a short time. (Equally well, $\phi(t)$ increases between the two curves V in figure 1(*a*).)

The trajectories return into the small-ϑ region from above (see figure 1(*a*)) and pass the plane $\alpha = \pi/2$ at a value of ϑ that is greater than $\vartheta_0 = 0$. Thus, the inner kinematic limit is a *limit-cycle* for the solution trajectory. ϑ oscillates between $\vartheta_{\max} = \vartheta_1$ and a $\vartheta_{\min}$ approaching 1 for $t \to \infty$ (nutations), α oscillates between values of $\alpha \to \pi/2 + \alpha_\infty$ and $\alpha \to \pi/2 - \alpha_\infty$ (oscillations about the position of stability $\alpha = \pi/2$, see § 2). In the mean over many periods of time we have $\langle \dot{\alpha} \rangle = 0$, that is, $\langle \dot{\varphi} \rangle = \langle \dot{\phi} \rangle$, though $|\dot{\varphi}|_{\max}$ is much greater than $|\dot{\phi}|_{\max}$. The angular velocity of mass centre about the z axis, $\mathrm{d}\phi/\mathrm{d}t$, reaches its maximum at large ϑ, where it is negative definite and nearly equal to $P\vartheta^2\omega$. The mass centre thus slowly gyrates about the z axis, the slower the greater ω. $\dot{\varphi} + \dot{\phi}$ is obviously always negative. This unidirectional motion in $\varphi + \phi$ corresponds to the rotational sense of the body, since (3.4) and (3.6*b*) show that both $\dot{\varphi}$ and $\dot{\phi}$ change sign if Ω_3 is multiplied by -1.

4. Stabilisation

The force f_r is given by

$$f_r = (g/2z^4)(\bar{\theta} \sin \alpha + 4r/z)$$

if $r/z \ll 1$ and $\bar{\theta} \ll 1$. We know the r/z term from § 2. As we saw in § 3, the $\bar{\theta}$ term is positive definite, that is, it enforces the effect of the r/z term. The mass centre thus moves away from the z axis, though slowly because $f_r \ll f_z$.

We can get a sign-changing f_r if we allow for a non-zero angle γ between the third axis and the dipole. To avoid the exertion of great centrifugal forces on the dipole, we separate the body into a flywheel and a platform, in the centre of which the dipole rests (figure 2). Let the old θ, φ, ψ be the Euler angles of the flywheel and θ', φ', ψ' those of the platform with reference to the axes of $O'XYZ$. We have $\theta' = \theta$ and $\varphi' = \varphi$, that is, there is only one additional degree of freedom, ψ'. The kinetic energy of motion of the platform relative to the mass centre is

$$\tfrac{1}{2}I_p(\dot{\theta}^2 + \dot{\varphi}^2 \sin^2 \theta) + \tfrac{1}{2}I_{p3}(\dot{\varphi} \cos \theta + \dot{\psi}')^2 \tag{4.1}$$

which must be added to the Lagrange function. The first term only enlarges I by I_p. The second term is small compared with the analogous term of the flywheel because we are interested in the case $\Omega_{p3} \ll \Omega_3$ and initially $\Omega_{p3} = 0$ so that the dipole moves slowly. (Ω_3 remains a constant of the motion.)

Let β be the azimuth angle of the dipole in the plane of the platform. The angles β and γ can be varied by inner forces. Let $O'x'y'z'$ be a system of rectangular axes fixed relative to the platform. The unit vector in the direction of the dipole is $\boldsymbol{e}'_M = (\sin \gamma \cos \beta, \sin \gamma \sin \beta, \cos \beta)$ in this frame of reference. Applying the transformation matrix for the transition from $O'x'y'z'$ to $O'XYZ$ in Euler angles (Whittaker 1955) to

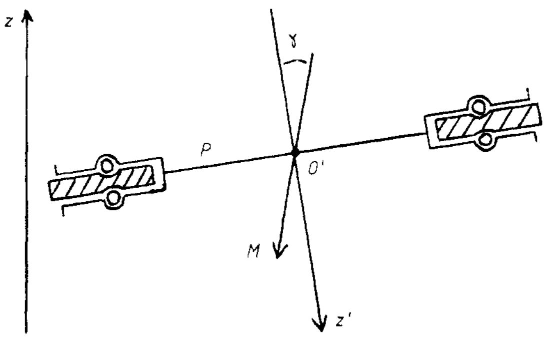

Figure 2. In the plane of a platform P of mass m_p a massive ring of mass m is rotating. The dipole M rests relative to P in the mass centre. Its direction is given by an angle γ and, in the $x'y'$ plane, an angle β. Friction in the bearings and mass of the dipole are neglected.

this vector, we find $\boldsymbol{e}_M$ with reference to the axes of $O'XYZ$. This then gives us for the potential energy:

$$-\boldsymbol{MH} = -MH_r \sin\gamma[\cos(\psi' - \varphi + \phi + \beta) - (1 + \cos\theta)\sin\alpha \sin(\psi' + \beta)] \\ -MH_z \sin\gamma \sin\theta \sin(\psi' + \beta) - (\boldsymbol{MH})_{\text{old}} \cos\gamma \tag{4.2}$$

where $(\boldsymbol{MH})_{\text{old}}$ is expression (1.3), H_r and H_z being given by (1.4).

The third term reproduces the equations of motion in § 3, the '$\cos\gamma$' only weakens the 'coupling constant' g. What is the meaning of the other terms for the motion in the $\theta\varphi\phi$ space?

The new expression for f_r at small r/z and $\bar{\theta}$ is

$$f_r = \frac{g}{2z^4}[\sin\gamma \cos(\psi' - \varphi + \phi + \beta) + \bar{\theta}\sin\alpha \cos\gamma + 4(r/z)\cos\gamma]. \tag{4.3}$$

We are interested in an f_r that can be kept at zero and therefore choose β such that

$$\psi' - \varphi + \phi + \beta = \pi. \tag{4.4}$$

We can now keep f_r equal to zero by varying $\sin\gamma$ if ω is so large that $\bar{\theta} \ll \sin\gamma$.

Because of the choice (4.4), the first term in (4.2) cannot contribute to the equations of motion of φ or ϕ. The second term is smaller than the corresponding term in $(\boldsymbol{MH})_{\text{old}}$ by a factor of $\bar{\theta}\gamma$ and can be neglected. The third term in (4.4) contributes to the equation of motion of $\bar{\theta}$ but not to the equation of motion of φ and ϕ. Instead of $h_r \sin\alpha$ on the RHS of (3.5*a*) and energy conservation (3.8), we now have $(h_r \cos\gamma - h_z \sin\gamma)\sin\alpha$, where we made use of (4.4). This is the torque about the line of nodes. A positive γ makes it smaller and thereby reduces $\bar{\theta}_{\max}$. An even larger γ makes the torque negative. Then the gradient field remains unchanged if only the sign of α is changed. In other words, the system behaves in $\theta\varphi\phi$ space in the same way as in the case of a positive torque, the only difference being that α oscillates about $-\pi/2$, the present minimum of the potential energy with respect to α.

Let us write down the equation of motion for Ω_{p3}, the angular velocity of the platform about its third axis

$$I_{p3}\frac{\mathrm{d}}{\mathrm{d}t}\Omega_{p3} = -MH_r \sin\gamma \sin(\psi' - \varphi + \phi + \beta) + MH_z(\sin\gamma)\bar{\theta}\cos(\psi' + \beta). \tag{4.5}$$

According to the choice (4.4), the first term disappears and the second term oscillates as $\bar{\theta}\cos\alpha$ about zero. In the mean, it thus makes a time-independent small contribution to $\langle\Omega_{p3}\rangle$. For $\cos\theta\sim-1$, however, we have $\Omega_{p3}\sim-\dot{\varphi}+\dot{\psi}'$, and then, it follows from (4.4) that $\Omega_{p3}\sim-\dot{\phi}-\dot{\beta}$, so that we can keep Ω_{p3} very small by varying β in the same way as ϕ varies, but opposite in sign.

4.1. *Neglecting the nutations*

If the rotational energy of the flywheel is sufficiently high and γ not too small, we can neglect the $\bar{\theta}$ terms in (4.3) and (4.5). Force f_r takes on the simple form

$$f_r=(g/2z^4)[-\sin\gamma+4(r/z)\cos\gamma]. \tag{4.6}$$

In this approximation, $\dot{\phi}=0$, $\Omega_{p3}=0$, planes $x'y'$ and XY are parallel, and the difference $\psi'-\varphi$ is their relative orientation. Let us call $\psi'-\varphi\equiv\delta$.

As has already been explained, the value (4.4), which made it possible to control f_r in (4.3) by varying γ, is an energetic maximum and, therefore, unstable. Small perturbations of ϕ or δ will induce an exponential change of the difference $\delta+\phi+\beta-\pi$. Expanding (4.5) and (3.5*c*) at $\delta+\phi+\beta-\pi\sim0$, one can show that this difference can, nevertheless, be kept in the vicinity of zero by appropriately varying β. In other words, the form (4.6) of f_r can be stabilised in this way.

On the other hand, $\delta-\phi$ cannot be kept constant. It will, in dependence of the perturbations, change if form (4.6) is stabilised. This change means that the $x'y'$ plane cannot be prevented from rotating to the XY plane while the mass centre simultaneously rotates about the z axis.

4.2. *Piloting the device*

Not only can the angle γ in (4.6) be used to keep f_r at zero, it can also be varied in such a way that the mass centre describes a certain path. Let us, for example, examine the case where the mass centre moves in the sector $r/z\ll1$, $\gamma\neq0$ is constant, and the azimuthal component of the dipole is parallel to $\boldsymbol{H}_r$, i.e. has a stable direction. Then

$$f_r=\frac{g}{2z^4}\left(\sin\gamma+4\frac{r}{z}\cos\gamma\right),\qquad f_z=\frac{g}{z^4}\left(\cos\gamma-2\frac{r}{z}\sin\gamma\right). \tag{4.7}$$

Let r_0 be very small. What is the asymptotic inclination of the trajectory? This problem can be solved by the methods in § 2.2. However, if there is a great initial linear momentum in the z direction, the problem can be solved directly†.

† Since then r/z will remain much smaller than $\tan\gamma$. We can neglect the r/z terms for all $\gamma\neq0$ and $\gamma\neq\pi/2$. As $\dot{z}_0$ is large, we can also neglect the increase of $\dot{z}_0$ in the field. The equation of the trajectory (2.6) takes the form

$$\mathrm{d}^2r/\mathrm{d}z^2=\tfrac{3}{4}C(\sin\gamma)z^{-4}-\tfrac{3}{2}C(\cos\gamma)z^{-4}\,\mathrm{d}r/\mathrm{d}z \tag{4.8}$$

where $C=\frac{2}{3}g/\dot{z}_0^2$ is the ratio of the initial potential energy at $\gamma=0$ to the kinetic energy (by supposition $C\ll1$). This is a linear first-order equation for $\mathrm{d}r/\mathrm{d}z$ which can be solved in general. We find

$$\mathrm{d}r/\mathrm{d}z\sim\tfrac{1}{4}C(\sin\gamma)(1-z^{-3}) \tag{4.10}$$

for $\mathrm{d}r/\mathrm{d}z_0=0$ in the approximation of $C(\cos\gamma)\ll1$. The asymptotic inclination is thus equal to $\frac{1}{4}C(\sin\gamma)$. If C decreases, this inclination can be kept constant by letting γ increase. If γ has become as large as $\pi/2$, however, $\mathrm{d}r/\mathrm{d}z$ decreases as C. At these large values of γ, the mass centre is primarily accelerated by the transverse force component f_r.

We can also give the trajectory $r(z)$ and calculate from (2.6) the dependence of γ on z that leads to this trajectory. Let us for instance examine the case where the device departs to infinity on a straight line $r = \lambda z$, $0 \leq \lambda < \infty$ (a spiral about the z axis for $\dot{\phi} \neq 0$), or comes down from infinity on such a line to the external field's centre. Because $d^2r/dz^2 = 0$, the differential equation (2.6) is particularly simple. Using the unapproximated expressions (1.4) of H_r and H_z in (4.7), we find

$$\tan \gamma = 2\lambda/(\lambda^2 - 1), \qquad 0 \geq \gamma \geq -\pi. \tag{4.9}$$

There is no dependence on position; for a straight line $r = \lambda(z-1)$, for example, γ would depend on the position.

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